ISRAEL JOURNAL OF MATHEMATICS, Vol. 42, Nos. 1-2, 1982

SOME GEOMETRICAL PROPERTIES OF BANACH SPACES OF POLYNOMIALS

BY

Y. GORDON[†] AND S. REISNER

ABSTRACT

We investigate the asymmetry, gl constants and best factorization estimates of the n-dimensional spaces of polynomials $H_p^n = \mathrm{span}\{e^{ikx}; k = 1, 2, \cdots, n\}$ equipped with the L_p norm for $1 \leqq p \leqq \infty$.

Notations

We use standard notations and terminology of Banach space theory mostly as it appears in [15].

In particular for a Banach space E, E^* is the dual space and for an operator $T: E \to F$, T^* denotes the adjoint of T. Notations for concepts related to the theory of Banach ideals are taken from [9], which, together with [21], can serve as a general reference on the subject.

The references for the theory of H_p spaces are [4] and [25]; also [18] can be used. Some special notations we use are: $\mathbf{T}$ is the unit circle $\{z; |z| = 1\}$ identified with the interval $[0, 2\pi]$ and equipped with the Lebesgue measure dt. e_k ($k = 0, \pm 1, \pm 2, \cdots$) are the functions on $\mathbf{T}$ defined by $e_k(t) = e^{ikt}$. H_p^n ($0 < p \leqq \infty$) is the n-dimensional subspace of H_p spanned by $\{e_k; k = 1, \cdots, n\}$. In parts of this paper we used for convenience a slightly different isometric version of H_p^n which is given there. Instead of an introduction for the whole paper we open each section with a short description of its contents.

We thank Y. Benyamini, W. B. Johnson and D. R. Lewis for some valuable discussions.

§1. The asymmetry constants of H_1^n and H_∞^n

It is well known that H_1 has local unconditional structure [17], moreover the Franklin basis forms an unconditional basis [24]. Recently Bourgain and

[†] Supported in part by NSF grant # MCS-8109561.
Received July 13, 1981 and in revised form October 21, 1981

Pelczynski proved that H_1^n can be uniformly embedded in H_1 so that its image is also uniformly complemented, hence the local unconditional structure constants of H_1^n are uniformly bounded. However it is still unknown whether H_1^n has a basis with uniformly bounded unconditional constants. We shall prove in this section that the asymmetry constants of H_p^n $(p = 1, \infty)$ tend to infinity with n; this implies that their symmetric basis constants tend to infinity as well.

Recall that if E is an n-dimensional Banach space, the asymmetry constant $s(E)$ is defined to be the least λ for which there exists a group of invertible operators G defined on E whose norms are at most λ and for which the only operators T on E which commute with each $g \in G$ are the scalar multiples of the identity 1_E on E ([6]). We shall denote by dg the normalized Haar measure associated with the compact group G. The main result of this section is:

THEOREM 1.1. *There exist positive constants* c_k $(k = 1, 2, 3)$ *such that for every* $n \geqq 1$

(1) $c_1 s^2(H_1^n) \geqq \gamma_1(H_1^n) \geqq c_2 \sqrt{\log n}$,

(2) $c_3 s^2(H_\infty^n) \geqq \log n$.

We need the following lemma which is proved independently in [11] and [2].

LEMMA 1.2. *For every* $\alpha > 0$ *there are* $\beta = \beta(\alpha)$, $0 < \beta < 1$, *and* $\gamma(\alpha) > 0$ *such that for every* n*-dimensional subspace* E *of* $L_1(0,1)$ *with* $d(E, l_1^n) \leqq \alpha$, E *contains a subspace* F *of dimension* m $(\geqq \beta n)$ *so that* $d(F, l_1^m) \leqq \alpha$ *and there is a projection from* L_1 *onto* F *with norm* $\leqq \gamma(\alpha)$.

The following lemma is an observation due to Lewis and Gordon:

LEMMA 1.3. *Given an* m*-dimensional Banach space* X, *let* $A \in L(l_p^n, X)$, $B \in L(X, l_p^n)$ *be operators for which* BA *is the identity on* l_p^n $(1 \leqq p \leqq \infty)$. *Then* $\gamma_p(X) \leqq mn^{-1} \|A\| \|B\| s^2(X)$.

PROOF. Let G be the group of operators on X such that $\|g\| \leqq s(X)$ for all $g \in G$. Define $\alpha : L_p(l_p^n, G, dg) \to X$ and $\beta : X \to L_p(l_p^n, G, dg)$ by

$$\alpha(f) = \int_G g^{-1} A f(g) dg \qquad (f \in L_p(l_p^n, G, dg))$$

and

$$(\beta x)(g) = Bgx \qquad (x \in X,\ g \in G).$$

Then, clearly $\|\beta\| \leqq \|B\| s(X)$, and

$$\|\alpha(f)\| \leqq \int_G \|g^{-1}\| \|A\| \|f(g)\| dg \leqq \|A\| s(X) \|f\|$$

so that $\|\alpha\| \leqq \|A\| s(X)$. Moreover, $\alpha\beta = \int_G g^{-1}ABg dg$, which implies that $\alpha\beta$ commutes with all elements of G, hence there exists a scalar λ for which $\alpha\beta = \lambda 1_X$. Therefore,

$$\lambda m = \operatorname{trace}(\alpha\beta) = \int_G \operatorname{trace}(g^{-1}ABg)dg = \int_G \operatorname{trace}(AB)dg = \operatorname{trace}(BA) = n$$

so that $\lambda = n/m$. Since $L_p(l_p^n, G, dg)$ is an L_p space, this concludes the proof. □

We now need theorem 7.10, chapter 10, [25] which states

THEOREM 1.4. *Let $P(z) = c_0 + c_1 z + \cdots + c_n z^n$, then*

$$\left(\frac{1}{n+1}\sum_{k=0}^{n} |P(e^{i2\pi k/(n+1)})|^p\right)^{1/p} \leqq A\left(\int_0^{2\pi} |P(e^{it})|^p dt\right)^{1/p} \quad (1 \leqq p \leqq \infty),$$

$$\left(\int_0^{2\pi} |P(e^{it})|^p\right)^{1/p} \leqq A_p\left(\frac{1}{n+1}\sum_{k=0}^{n} |P(e^{i2\pi k/(n+1)})|^p\right)^{1/p} \quad (1 < p < \infty),$$

where A, A_p are constants independent of n.

PROOF OF THEOREM 1.1. For convenience we shall prove the theorem for the spaces $H_p^{2(2n+1)} = \operatorname{span}\{e^{ikt}; -2n \leqq k \leqq 2n+1\}$ $(p = 1, \infty)$, and define the polynomials

$$P_0(t) = (2n+1)^{-1}\sum_{k=-n}^{n} e^{ikt} \quad \text{and} \quad P_k(t) = P_0\left(t - \frac{2\pi k}{2n+1}\right)$$

$(k = 0, \pm 1, \cdots, \pm n)$. It is easy to see that $\int_0^{2\pi} P_k^2(t)dt = 2\pi/(2n+1)$. Let $P(t) = \sum_{k=-n}^{n} c_k P_k^2(t)$ where c_k are arbitrary scalars. Then $P(t) \in H_1^{2(2n+1)}$ and $\|P\|_{L_1} \leqq (1/(2n+1))\sum |c_k|$.

On the other hand $P_k(2\pi j/(2n+1)) = \delta_{k,j}$, hence $P(2\pi k/(2n+1)) = c_k$, and by Theorem 1.4 applied to the space $H_1^{2(2n+1)}$

$$\begin{aligned} 2\pi A \|P\|_{L_1} &\geqq \frac{1}{2(2n+1)}\sum_{k=0}^{4n+1} \left|P\left(\frac{2\pi k}{2(2n+1)}\right)\right| \\ &\geqq \frac{1}{2(2n+1)}\sum_{j=0}^{2n} \left|P\left(\frac{2\pi j}{2n+1}\right)\right| = \frac{1}{2(2n+1)}\sum_{j=-n}^{n} \left|P\left(\frac{2\pi j}{2n+1}\right)\right| \\ &= \frac{1}{2(2n+1)}\sum_{j=-n}^{n} |c_j|, \end{aligned}$$

therefore $\operatorname{span}\{P_k^2; k = 0, \pm 1, \cdots, \pm n\}$ is a subspace of $H_1^{2(2n+1)}$ which is uniformly isomorphic to l_1^{2n+1}. By Lemma 1.2 $H_1^{2(2n+1)}$ contains a subspace of dimension N $(\geqq \gamma n)$ uniformly isomorphic to l_1^N and uniformly complemented in $L_1[0, 2\pi]$, hence also in $H_1^{2(2n+1)}$. By Lemma 1.3,

$$s^2(H_1^{2(2n+1)}) \geqq \frac{cN}{2(2n+1)} \gamma_1(H_1^{2(2n+1)}).$$

It is well known that H_1^n contains $l_2^{[\log n]}$ uniformly complemented (supported on a lacunary set), thus the identity on $l_2^{[\log n]}$ can be factored as

$$l_2^{[\log n]} \xrightarrow{A} H_1^{2(2n+1)} \xrightarrow{B} L_1 \xrightarrow{C} H_1^{2(2n+1)} \xrightarrow{D} l_2^{[\log n]}$$

where $\|A\|\|D\| \leqq \text{const.}$, and $\|B\|\|C\| = \gamma_1(H_1^{2(2n+1)})$. By Grothendieck $\pi_1(DC) \leqq \|DC\| K_G$, hence

$$\|A\|\|D\|\|B\|\|C\| \geqq \|BA\|\|DC\| \geqq K_G^{-1}\|BA\|\pi_1(DC) \geqq K_G^{-1}\pi_2(l_2^{[\log n]}) \sim \sqrt{\log n}$$

(the last equivalence is by [6]). This concludes the proof of (1).

To prove (2), we can use a simpler argument. First, $H_\infty^{2(2n+1)}$ contains a subspace of dimension $2n+1$ uniformly isomorphic to l_∞^{2n+1} (see [3] or notice that

$$P_k(t) = \frac{(-1)^k \sin\left(n+\frac{1}{2}\right)t}{(2n+1)\sin\left(\frac{t}{2} - \frac{k\pi}{2n+1}\right)},$$

and therefore

$$\|P\|_\infty = \left\| \sum_{k=-n}^{n} c_k P_k^2 \right\|_\infty \leqq \left(\max_k |c_k| \right) \max_t \sum_{k=-n}^{n} P_k^2(t) \leqq c \max_k |c_k|$$

(where c does not depend on n). Conversely,

$$\|P\|_\infty \geqq \left| \sum_{k=-n}^{n} c_k P_k^2 \left(\frac{2\pi l}{2n+1}\right) \right| = |c_l| \qquad \text{for all } l).$$

Now, by Hahn–Banach l_∞^{2n+1} factors uniformly through $H_\infty^{2(2n+1)}$, therefore we can apply Lemma 1.3 with $p=\infty$ and obtain that $s^2(H_\infty^{2(2n+1)}) \geqq c\gamma_\infty(H^{2(2n+1)}) \sim \log n$ (the last equivalence is a well known fact).

Added Remark. We observed that Lemma 1.2 is not needed since span$\{P_k^2\}$ is naturally complemented in $H_1^{2(2n+1)}$. Moreover, by using another basis one can embed l_1^m in H_1^n, and l_∞^m in H_∞^n uniformly complemented, with $m = \alpha n$, for any $0 < \alpha < 1$, with constants depending only on α.

§2. L_1-factorization of operators $T: H_\infty^n \to l_2$

For a Banach space E we define the G.L. constant of E, gl(E), as

$$\mathrm{gl}(E) = \sup \left\{ \frac{\gamma_1(T)}{\pi_1(T)} \,\middle|\, T : E \to H,\ H \text{ a Hilbert space} \right\}.$$

We recall that

$$\mathrm{gl}(E) \leqq \chi(E) \leqq \mathrm{u.c.b}(E)$$

where $\chi(E)$ is the (G.L.) l.u.s.t. constant of E and u.c.b the unconditional basis constant of E ([18]). It is known that $\mathrm{gl}(H_\infty) = \infty$ (see e.g. [18]). We shall prove here that $\mathrm{gl}(H_\infty^n) \to \infty$ as $n \to \infty$, namely:

PROPOSITION 2.1. *$\mathrm{gl}(H_\infty^n) \geqq c\sqrt{\log n}$ where c is an absolute positive constant.*

Proposition 2.1 is a simple consequence of an inequality of Kwapien–Pelczynski [14]. We shall also give here a different proof of a somewhat stronger inequality from which the result of [14] follows (as Corollary 2.3 (b)). We need some prerequisites.

(a) Let M_n be the subspace of $L_1(\mathbf{T})$ defined by

$$M_n = \{ f \in L_1(\mathbf{T}) \mid \hat{f}(k) = 0 \text{ for } 1 \leqq k \leqq n \}.$$

We identify $(H_\infty^n)^*$ with $L_1(\mathbf{T})/M_n$ by

$$\langle g, [f] \rangle = \frac{1}{2\pi} \int_0^{2\pi} g \bar{f} dt; \qquad g \in H_\infty^n, \quad f \in L_1(\mathbf{T})$$

($[f]$ — the equivalence class of f). From Kolmogorov's inequality (cf. [4]) it follows that for every $0 < p < 1$ there is a constant d_p independent of n, such that the operator

$$R_n : L_1(\mathbf{T})/M_n \to L_p(\mathbf{T})$$

which is defined by

$$R_n[e_k] = e_k \qquad (k = 1, \cdots, n)$$

satisfies

(1) $$\|R_n\| < d_p.$$

(b) Let E, F be finite dimensional Banach spaces and $T : E \to F$ a linear operator. Let $J : F \to L_p(\Omega, \Sigma, \mu)$ $(0 < p < \infty)$ be a bounded linear operator. There is a unique (up to equality a.e.) measurable function $\phi : \Omega \to E^*$ which satisfies for all $x \in E$:

$$(JTx)(\cdot) = \langle x, \phi(\cdot) \rangle \quad \text{a.e.}$$

From [23] and [13] it follows that

$$\|\phi\|_{L_p(\mu,E^*)} \leqq \|J\|\,\pi_p(T^*). \tag{2}$$

PROPOSITION 2.2. *For every $0<p<1$ there are $A_p, B_p>0$ such that for all n, if $\{f_k\}_{k=1}^n$ is a basis for H_∞^n, $\{f_k^*\}_{k=1}^n$ are the coefficient functionals of $\{f_k\}$ and $\{g_j\}_{j=1}^m$ is an orthonormal basis of l_2^m then for every $T: H_\infty^n \to l_2^m$ which is represented by a matrix $(t_{jk})_{1\leqq j\leqq m, 1\leqq k\leqq n}$ (i.e., $Tf_k = \sum_{j=1}^m t_{jk}g_j$) holds*

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{j=1}^m\left|\sum_{k=1}^n t_{jk}(R_nf_k^*)(t)\right|^2\right]^{p/2}dt\right\}^{1/p} \leqq A_p\pi_p(T) \leqq B_p\gamma_1(T). \tag{3}$$

PROOF. Fix $0<p<1$. From [14], theorem 91, it follows that there is a constant b_p such that for all $T: H_\infty^n \to l_2$

$$\pi_p(T) \leqq b_p\gamma_1(T). \tag{4}$$

Let $\phi: \mathbf{T}\to l_2^m$ be the function from (b), relative to $E = l_2^m$, $F=(H_\infty^m)^* = L_1(\mathbf{T})/M_n$, $(\Omega,\mu) = (\mathbf{T}, dt/2\pi)$ and $J = R_n$ where T^* replaces T in (b).

$$l_2^m \xrightarrow{T^*} L_1(\mathbf{T})/M_n \xrightarrow{R_n} L_p(\mathbf{T})$$

i.e. $(R_nT^*x)(t) = \langle\phi(t), x\rangle$ a.e. for all $x\in l_2^m$. We have, by (2), (1) and (4),

$$\|\phi\|_{L_p(\mathbf{T}, l_2^m)} \leqq d_p\pi_p(T) \leqq b_pd_p\gamma_1(T). \tag{5}$$

Let $\phi(t) = \sum_{j=1}^m a_j(t)g_j$ be the representation of ϕ in the basis $\{g_j\}$. We have

$$(R_nT^*g_j)(t) = \langle\phi(t), g_j\rangle = a_j(t).$$

On the other hand, representing $(H_\infty^n)^*$ in the basis $\{f_k^*\}$ we get

$$(R_nT^*g_j)(t) = \sum_{k=1}^n t_{jk}(R_nf_k^*)(t)$$

hence

$$a_j(t) = \sum_{k=1}^n t_{jk}(R_nf_k^*)(t) \tag{6}$$

which, together with (5), yields (3). □

COROLLARY 2.3. *Let T be as in Proposition 2.2 with $f_k = e_k$, $f_k^* = [e_k]$, $k = 1,\cdots,n$.*

(i) *Let $0<p<1$ and K_0 be a number such that for all $i\leqq j\leqq m$ holds*

$$\left[\sum_{k=1}^n |t_{jk}|^2\right]^{1/2} \leqq K_0\left\|\sum_{k=1}^n t_{jk}e_k\right\|_{L_p(\mathbf{T})}.$$

Then

(a) $$\nu_1(T) \leqq \Big[\sum_{j,k} |t_{jk}|^2\Big]^{1/2} \leqq K_0 A_p \pi_p(T) \leqq K_0 B_p \gamma_1(T).$$

In particular, if $\Lambda = (\lambda_j)_{j=1}^n$ *is a multiplier from* H_∞^n *into* l_2^n *then*

(b) $$\nu_1(\Lambda) \leqq \Big[\sum_{j=1}^n |\lambda_j|^2\Big]^{1/2} \leqq cK_0 B_p \gamma_1(\Lambda)$$

where c *is an absolute constant.*

(ii) *Let* $\varepsilon = (\varepsilon_k)_{k=1}^n$, $\varepsilon_k = \pm 1$ *and define* $T_\varepsilon : H_\infty^n \to l_2^m$ *by the matrix* $(\varepsilon_k t_{jk})$. *Then*

(c) $$\nu_1(T) \leqq \Big[\sum_{j,k} |t_{jk}|^2\Big]^{1/2} \leqq cAv_\varepsilon \gamma_1(T_\varepsilon)$$

where c *is an absolute constant.*

Proof. The inequality $\nu_1(T) \leqq [\sum_{j,k} |t_{jk}|^2]^{1/2}$ is a simple consequence of the following factorization of T

$$\begin{array}{ccccc} H_\infty^n & & \xrightarrow{\quad T \quad} & & H_2^m \\ I \big\downarrow & & & & \big\downarrow \tilde{T} \\ C(\mathbf{T}) & \xrightarrow{J} & L_2(\mathbf{T}) & \xrightarrow{P} & H_2^m \end{array}$$

Here we identify l_2^m with the subspace H_2^m of H_2, spanned by $\{e_j\}_{j=1}^m$. I is the inclusion map, J the formal inclusion, P is the natural projection and $\tilde{T}$ the operator in H_2^m defined by the matrix (t_{ij}). By [19] we have

$$\nu_1(T) \leqq \pi_2(\tilde{T})\pi_2(J) = \mathrm{hs}(\tilde{T}) = \Big[\sum_{j,k} |t_{jk}|^2\Big]^{1/2}$$

(hs — Hilbert Schmidt norm). For the right hand side inequality in (a) we use (3) and Minkowski's inequality which yields

(7) $$\begin{aligned} &\left\{\sum_{j=1}^m \left[\frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{k=1}^n t_{jk} e_k(t)\right|^p dt\right]^{2/p}\right\}^{1/2} \\ &\leqq \left\{\frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{j=1}^m \left|\sum_{k=1}^n t_{jk} e_k(t)\right|^2\right]^{p/2} dt\right\}^{1/p}. \end{aligned}$$

To prove (c), we replace in (7) (t_{jk}) by $(\varepsilon_k t_{jk})$, average over ε and use the Khintchine–Kahane inequality. □

REMARKS. Inequality (b) was proved in [14] in the infinite dimensional case and the same proof applies to the finite dimensional case.

Proposition 2.2 and Corollary 2.3 can be adapted to the infinite dimensional case. A slight modification of the preceding proof yields the following generalization of a result of [14]. Let $H_1^0 = \{f \in L_1(\mathbf{T}) \mid \hat{f}(n) = 0 \text{ for } n \geqq 0\}$, for $[\hat{g}] \in L_1(\mathbf{T})/H_1^0$, $[\hat{g}](n)$ is well defined for all $n \geqq 0$ by $[\hat{g}](n) = \hat{g}(n)$. Also, the operator $R : L_1(\mathbf{T})/H_1^0 \to L_p(\mathbf{T})$ $(0 < p < 1)$ is well defined by $R[e_k] = e_k$ $(0 \leqq k)$ and bounded by a constant K_p.

Let $T : l_2 \to L_1/H_1^0$ and denote

$$t_{n,k} = \hat{T}(n,k) = (\widehat{Tg_n})(k)$$

$(n, k = 0, 1, 2, \cdots, g_n$ — u.v.b. in $l_2)$.

PROPOSITION 2.4. (i) *For every* $0 < p < 1$ *there are* A_p, B_p *such that*

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} \left[\sum_n |(RTg_n)(t)|^2 \right]^{p/2} dt \right\}^{1/p} \leqq A_p \pi_p(T^*) \leqq B_p \gamma_\infty(T).$$

(ii) *Assume that there is a constant c such that for every n,*

$$\varliminf_{v \to 1^-} \left\| \sum_{k=1}^{\infty} t_{nk} v^k e_k \right\|_{L_p(\mathbf{T})} \geqq c \left(\sum_k |t_{nk}|^2 \right)^{1/2}$$

(a particular case — when all columns of the matrix (t_{nk}) *are supported on* Λ_2 *sets with uniformly bounded constant), then the following are equivalent:*

(1) *T factors through an* L_∞ *space.*

(2) T^* *is nuclear.*

(3) T^* *is* 0*-absolutely summing.*

(4) $[\sum_{n,k} |t_{nk}|^2]^{1/2} < \infty$.

REMARK. Recently, Kisliakov proved and used in the proof of theorem I of [12] an inequality (lemma I) which is an easy corollary of Proposition 2.4.

We now turn to the

PROOF OF PROPOSITION 2.1. We bring two types of examples of operators

$$T : H_\infty^n \to H_2^n$$

for which

$$\frac{\gamma_1(T)}{\pi_1(T)} \geqq c\sqrt{\log n}. \tag{8}$$

EXAMPLE 1. $\Lambda : H_\infty^n \to H_2^n$ is the multiplier $\Lambda = (\lambda_j)_{j=1}^n$, $\lambda_j = j^{-1/2}$ $(j = 1, \cdots, n)$. Λ has a factorization

$$H_\infty^n \xrightarrow{J} H_1^n \xrightarrow{\tilde{\Lambda}} H_2^n$$

where J is the formal identity and $\tilde{\Lambda}$ is the multiplier defined by (λ_j). We have

$$\pi_1(\Lambda) \leqq \|\tilde{\Lambda}\| \leqq K \sup_{1 \leqq m \leqq n} \left[\frac{1}{m}\left(\sum_{j=1}^{m} j^2 j^{-1}\right)^{1/2}\right] \leqq K$$

(cf. [4] theorem 6.7), K independent of n. On the other hand, from Corollary 2.3 (b) we get

$$\gamma_1(\Lambda) \geqq \frac{1}{B_p}\left(\sum_{j=1}^{n} j^{-1}\right)^{1/2} \sim \frac{1}{B_p}\sqrt{\log n}.$$

EXAMPLE 2. $\Lambda : H_\infty^{2^n} \to H_2^{2^n}$ the Paley operator, defined as the multiplier $\Lambda = (\lambda_j)$, $\lambda_j = 1$ for $j = 2^k$ $(k = 0, \cdots, n)$, $\lambda_j = 0$ for $j \neq 2^k$. By Paley's theorem we get $\pi_1(\Lambda) \leqq \|\tilde{\Lambda}\| \leqq K$, while by Corollary 2.3

$$\gamma_1(\Lambda) \geqq \frac{1}{B_p}\sqrt{n}.$$

The proofs of Theorem 1.1 and Proposition 2.1 can be applied to show

THEOREM 2.5. *Let $p = 1, \infty$ and E_p be an m-dimensional space for which $H_p \supseteq E_p \supseteq H_p^n$ (the inclusions here are the natural ones). Then*

(a) $\mathrm{cgl}(E_\infty) \geqq \sqrt{\log n}$.

(b) $\mathrm{cs}^2(E_p) \geqq m^{-1} n \sqrt{\log n}$ $(p = 1, \infty)$.

PROOF. (a) Let $S : H_\infty \to H_2$ be the operator defined by $S(f) = \sum_{k=1}^{[\log_2 n]} e_{2^k}(t) \int_0^{2\pi} f(s)\overline{e_{2^k}(s)}\,ds$. Factoring $S|_{E_\infty} : E_\infty \xrightarrow[J]{} E_1 \xrightarrow[S|_{E_1}]{} H_2$, where J is the identity, shows that

$$\pi_1(S|_{E_\infty}) \leqq \pi_1(J)\|S|_{E_1}\| = \|S|_{E_1}\| \leqq c \qquad \text{(a constant)}.$$

On the other hand, since $E_\infty \supseteq H_\infty^n$, by Example 2 above $\gamma_1(S|_{E_\infty}) \geqq \gamma_1(S|_{H_\infty^n}) \geqq c\sqrt{\log n}$, which proves (a).

(b) The proof for $p = 1$ is identical to that of Theorem 1.1 (1), using the full strength of Lemma 1.2 which implies that E_1 contains $l_1^{[\gamma n]}$ uniformly complemented $(0 < \gamma < 1$ independent of n and $E_1)$, and then applying Lemma 1.3 together with the fact that E_1 contains $l_2^{[\log_2 n]}$ uniformly complemented and therefore

$$\gamma_1(E_1) \geqq c\gamma_1(l_2^{[\log_2 n]}) \geqq c\sqrt{\log n}.$$

In the case $p=\infty$, we note first that E_∞ contains $l_\infty^{[n/2]}$ uniformly complemented, and since L_∞ is a Banach lattice, by [8] and (a) $\gamma_\infty(E_\infty) \geqq \mathrm{gl}(E_\infty) \geqq c\sqrt{\log n}$, thus Lemma 1.3 concludes the proof.

REMARKS. (1) We do not know if the estimate of (b) for $p=\infty$ can be improved to $\mathrm{cs}^2(E_\infty) \geqq m^{-1} n \log n$ (which is true if $E_\infty = H_\infty^n$).

(2) Theorem 2.5 is no longer true if it is only assumed that $E_p \supset H_p^n$ isomorphically ($p = 1, \infty$), because by [25] (ch. X, theorem 7.28) and Theorem 1.4 above l_p^{4n} contains H_p^n uniformly for $p=1$ and ∞.

§3. Best factorization estimates for H_p^n spaces

By Theorem 1.4, if we take $\{P_k\}_{k=-n}^{n}$ to be the basis in H_p^{2n+1}, then $d(H_p^{2n+1}, l_p^{2n+1}) \leqq c_p$ if $1<p<\infty$, and $d(H_p^{2n+1}, l_p^{2n+1}) \leqq c\log(n+1)$ if $p=1$ or ∞. Since $d(l_p^n, l_q^n)$ is known for all values of p, q [10], it is easy to get trivial estimates for $d(H_p^{2n+1}, H_q^{2n+1})$, which are also asymptotically exact in n when $1<q<p<\infty$. We shall derive here some better and more general estimates in the non-trivial cases where p or q is in $\{1,\infty\}$.

Given Banach spaces E, F and G, let $\mathcal{F}(E,F,G)$ denote the quantity $\inf \|A\|\,\|B\|\,\|C\|$, where the infimum ranges over all $A \in L(E,F)$, $B \in L(F,G)$, $C \in L(G,E)$ for which $CBA = 1_E$. If $F=G$, we write $\mathcal{F}(E,F) = \mathcal{F}(E,F,F)$, and clearly $d(E,F) = \mathcal{F}(E,F)$ if E and F are isomorphic.

If we denote by $P_p^{(n)}$ the natural projection of L_p onto H_p^n, it is well known that $\|P_p^{(n)}\| \leqq c_p$ for $1<p<\infty$ [4], and $\|P_p^{(n)}\| \leqq c\log(n+1)$ if $p=1$ or ∞, thus $\mathcal{F}(H_p^n, H_p) \leqq c_p$ if $1<p<\infty$, and $\mathcal{F}(H_\infty^n, H_\infty) \leqq c\log(n+1)$. Bourgain and Pelczynski recently proved $\mathcal{F}(H_p^n, H_p) \leqq C_p$ for all $1 \leqq p \leqq \infty$.

Throughout we denote by c, c_1, c_2, etc., constants, and by c_p constants which depend on p; the same letter may denote different constants in some cases.

We start with the following straightforward lemma whose proof is omitted.

LEMMA 3.1. *Let $I_{p,q}^{(n)}: H_p^n \to H_q^n$ be the natural injection, then $\|I_{p,q}^{(n)}\| \sim \max\{1, n^{1/p-1/q}\}$ for every $p,q \in [1,\infty]$.*

If T is an operator on l_2^n into some Banach space, $l(T)$ will denote $(\mathbf{E}_\omega \|\sum_{i=1}^n g_i(\omega) T(e_i)\|^2)^{1/2}$, where $\{g_i(\omega)\}_1^n$ is a sequence of standard independent normalized Gaussian variables, and $\{e_i\}_1^n$ any orthonormal basis for l_2^n (see [1] for details and references).

LEMMA 3.2. *If $1 \leqq p < \infty$, then for all $n > 1$*

$$l(I_{2,\infty}^{(n)*-1}) \sim l(I_{2,p}^{(n)}) \sim \sqrt{n},$$

and

$$l(I_{2,\infty}^{(n)}) \sim \sqrt{n \log n}.$$

PROOF. For convenience we replace n by $2n+1$ and denote by $Q_k^{(p)} = \sqrt{2n+1}\,P_k$ $(k = 0, \pm 1, \cdots, \pm n)$ the basis for the space H_p^{2n+1}. Let L_p^{2n+1} be the L_p space of dimension $2n+1$ with the normalized measure that assigns mass $(2n+1)^{-1}$ to each basis element $e_k^{(p)}$, $k = 0, \pm 1, \cdots, \pm n$.

If $T : H_p^{2n+1} \to L_p^{2n+1}$ is the basis to basis map $Q_k^{(p)} \to e_k^{(p)}$, then by Theorem 1.4 both $\|T\|$ and $\|T^{-1}\|$ are uniformly bounded with respect to n for every $1 < p < \infty$. Thus the estimates for $1 < p < \infty$ follow from the same estimates for L_p^{2n+1} which are easy to verify (see e.g. [1]).

If $p = 1$, using the well known properties of the Gaussian variables we have

$$\begin{aligned} l(I_{2,1}^{(n)}) &= \left(\mathbf{E}\left\| \sum_{k=1}^{n} g_k e^{ikt} \right\|_{H_1^n}^2\right)^{1/2} \\ &\sim \mathbf{E}\left\| \sum g_k e^{ikt} \right\|_{H_1^n} = \frac{1}{2\pi}\int_0^{2\pi} \left(\mathbf{E}\left| \sum g_k e^{ikt} \right|\right) dt \\ &\sim \frac{1}{2\pi}\int_0^{2\pi} \left(\sum |e^{ikt}|^2\right)^{\frac{1}{2}} dt = \sqrt{n}. \end{aligned}$$

The case $p = \infty$ needs some additional computations. Since $Q_k = \sqrt{2n+1}\,P_k$ $(k = 0, \pm 1, \cdots, \pm n)$ is an orthonormal basis for H_2^{2n+1} and the quantities

$$l(I_{2,\infty}^{(2n+1)}) \sim E_\omega \left\| \sum_{k=-n}^{n} g_k(\omega) e_k \right\|_{H_\infty^{2n+1}}$$

are both independent of the choice of the orthonormal basis $\{e_k\}_{k=-n}^{n}$ in H_2^{2n+1}, therefore it is enough to prove $E_\omega \|\Sigma_{k=-n}^{n} g_k(\omega) P_k\|_\infty \sim \sqrt{\log n}$.

Since $P_k(2\pi j/(2n+1)) = \delta_{k,j}$, it follows that $E\|\Sigma g_k P_k\|_\infty \geqq E(\max_j |g_j|) \sim \sqrt{\log n}$.

To prove the converse inequality, let $A = [\|\Sigma g_k(\omega) P_k\|_\infty > \alpha]$, where α will be chosen later. Then

$$\begin{aligned} \mathbf{E}\left\| \sum g_k P_k \right\|_\infty &\leqq \alpha + \int_A \left\| \sum g_k P_k \right\|_\infty \mathcal{P}(dw) \\ &\leqq \alpha + \sum \int_A |g_k(\omega)| \mathcal{P}(dw) \leqq \alpha + (2n+1)\sqrt{\mathcal{P}(A)}. \end{aligned}$$

Let $t_k = k\pi/4n$ ($k = 0, \pm 1, \cdots, \pm 4n$). By theorem 7.28 [25] there exists $c_1 > 0$ (independent of n) for which $\|P\|_{H_\infty^{2n+1}} \leqq c_1 \max_k |P(t_k)|$ for every $P \in H_\infty^{2n+1}$. Therefore

$$\mathcal{P}(A) \leqq \mathcal{P}\left(\left[\max_i \left|\sum_{k=-n}^{n} g_k(\omega) P_k(t_i)\right| > \alpha/c_1\right]\right)$$

$$\leqq 10n \max_i \mathcal{P}\left(\left[\left|\sum_{k=-n}^{n} g_k(\omega) P_k(t_i)\right| > \alpha/c_1\right]\right)$$

$$\leqq 10n \max_t \mathcal{P}\left(\left[\left|\sum_{k=-n}^{n} g_k(\omega) P_k(t)\right| > \alpha/c_1\right]\right).$$

Due to the symmetry of the expression in the intervals

$$I_k = \left[\frac{(2k-1)\pi}{2n+1}, \frac{(2k+1)\pi}{2n+1}\right] \qquad (k = 0, \pm 1, \cdots, \pm n)$$

the maximum is achieved at $t_0 \in I_0$. Using the identity

$$P_k(t) = P_0\left(t - \frac{2\pi k}{2n+1}\right) = \frac{(-1)^k \sin\left(n + \frac{1}{2}\right) t}{(2n+1)\sin\left(\frac{t}{2} - \frac{k\pi}{2n+1}\right)}$$

it follows that $|P_0(t)| \leqq 1$ and $|P_k(t)| \leqq c_2/|k|$ for all $1 \leqq |k| \leqq n$ and $t \in I_0$, hence by the contraction principle

$$\mathcal{P}\left(\left[\left|\sum g_k(\omega) P_k(t_0)\right| > \alpha/c_1\right]\right) \leqq \mathcal{P}\left(\left[\left|g_0(\omega) + \sum_{1 \leqq |k| \leqq n} \frac{g_k(\omega)}{k}\right| > \alpha/c_1c_2\right]\right)$$

$$\leqq \mathcal{P}([|g_0(\omega)| > \alpha/2c_1c_2]) + \mathcal{P}\left(\left[\left|\sum_{1 \leqq |k| \leqq n} \frac{g_k(\omega)}{k}\right| > \alpha/2c_1c_2\right]\right).$$

By Tchebychev's inequality

$$\mathcal{P}\left(\left[\left|\sum_{1 \leqq |k| \leqq n} \frac{g_k}{k}\right| > c_3\alpha\right]\right) \leqq 2e^{-c_4\alpha^2/\Sigma 1/k^2} \leqq 2e^{-c_5\alpha^2}$$

and so

$$\mathcal{P}(A) \leqq c_6 n e^{-c_7\alpha^2}.$$

Therefore, $(2n+1)^2 \mathcal{P}(A) \leqq c_6(2n+1)^2 n e^{-c_7\alpha^2}$ which shall tend to zero if we choose $\alpha = 2c_7^{-1/2}\sqrt{\log(n+1)}$. This completes the proof of $l(I_{2,\infty}^{(n)}) \sim \sqrt{n \log n}$.

Since $(H_\infty^n)^*$ is identified with $L_1(\mathbf{T})/M_n$, therefore

$$l(I_{2,\infty}^{(n)*-1}) = \left(\mathbf{E}\left\|\sum g_k e^{ikt}\right\|^2_{(H_\infty^n)^*}\right)^{1/2} \leqq l(I_{2,1}^{(n)}) \sim \sqrt{n}.$$

On the other hand it follows from the boundedness of the natural operator $R_n : L_1(\mathbf{T})/M_n \to L_{1/2}(\mathbf{T})$ and Kahane's inequality that

$$l(I_{2,\infty}^{(n)*-1}) \sim \mathbf{E}\left\|\sum_1^n g_k e^{ikx}\right\|_{(H_\infty^n)^*} \geqq c\mathbf{E}\left\|\sum_1^n g_k e^{ikx}\right\|_{H_{1/2}^n}$$

$$\sim \left(\mathbf{E}\left\|\sum_1^n g_k e^{ikx}\right\|^{1/2}_{H_{1/2}^n}\right)^2 = \left(\int_{\mathbf{T}} \mathbf{E}\left|\sum g_k e^{ikx}\right|^{1/2} dm\right)^2$$

$$\sim \left(\int_{\mathbf{T}} \left(\mathbf{E}\left|\sum g_k e^{ikx}\right|^2\right)^{1/4} dm\right)^2 = \sqrt{n}. \qquad \square$$

If $\{x_i\}_1^n$ is a finite sequence of vectors in a Banach space X, we denote $\varepsilon_2(x_i) = \sup(\|\sum t_i x_i\|/(\sum|t_i|^2)^{1/2})$, which is also the norm of the map $l_2^n \to X$ induced by $e_i \to x_i$.

We shall need the following theorem proved in [1].

THEOREM 3.3. *Let E, F, G be Banach spaces, $F \subseteq G$, and suppose $\{e_i, e_i^*\}_{i=1}^n$ is a basis with associated coefficient functionals for E, and $\{f_j, f_j^*\}_{j=1}^m$ is a biorthogonal sequence in G where $\{f_j\}_{j=1}^m \subset F$ and $m \geqq n$. Then*

$$\mathcal{F}(E,F,G) \leqq cm^{-1}\left\{\varepsilon_2(e_i^*)\mathbf{E}_\omega\left\|\sum_{j=1}^n g_j(\omega)f_j\right\| + \varepsilon_2(f_j)\mathbf{E}_\omega\left\|\sum_{i=1}^n g_i(\omega)e_i^*\right\|\right\}$$

$$\cdot\left\{\varepsilon_2(e_i)\mathbf{E}_\omega\left\|\sum_{j=1}^m g_j(\omega)f_j^*\right\| + \varepsilon_2(f_j^*)\mathbf{E}_\omega\left\|\sum_{i=1}^n g_i(\omega)e_i\right\|\right\}.$$

THEOREM 3.4. *For every Banach space X, Y for which $H_\infty^n \subseteq X \subseteq L_\infty$, $H_1^n \subseteq Y \subseteq L_1$, and every $1 < p < \infty$*

(i) $\mathcal{F}(H_p^n, H_\infty^n, X) \sim d(H_p^n, H_\infty^n) \sim \min\{n^{1/p}, n^{1/2}\}$,

(ii) $\mathcal{F}(H_p^n, H_1^n, Y) \sim d(H_p^n, H_1^n) \sim \min\{n^{1/2}, n^{1-1/p}\}$.

PROOF. (i) We factor the identity on H_p^n as follows:

$$H_p^n \xrightarrow[I_{p,\infty}^{(n)}]{} H_\infty^n \xrightarrow[j]{} X \xrightarrow[I]{} L_p \xrightarrow[P_p^{(n)}]{} H_p^n$$

where j is the inclusion, I is the restriction to X of the injection $L_\infty \to L_p$. Using the estimates of Lemma 3.1

$$d(H_p^n, H_\infty^n) \leqq \mathcal{F}(H_p^n, H_\infty^n, X) \leqq \|I_{p,\infty}^{(n)}\|\,\|j\|\,\|I\|\,\|P_p^{(n)}\| \leqq c_p n^{1/p}.$$

Conversely, if we denote by $\alpha_p(E)$ ($\beta_p(E)$) the type p (cotype p) constants of a Banach space E, then using the facts that H^n_∞ contains $l_\infty^{[n/2]}$ uniformly, and that L_p has cotype p if $p \geqq 2$, it follows that $d(H^n_p, H^n_\infty) \geqq \beta_p(H^n_\infty)/\beta_p(H^n_p) \geqq c_p\beta_p(l_\infty^{[n/2]}) \sim n^{1/p}$. Thus (i) is proved for $p \geqq 2$.

Let $1 < p \leqq 2$ and $\{e^{ikt}, e^{ikt}\}_{k=1}^n$ be the basis, and biorthogonal sequence, in the spaces H^n_p and X respectively. Applying the estimates of Lemmas 3.1 and 3.2 we get

$$\varepsilon_2(\{e^{ikt}\} \subset H^n_p) = \| I^{(n)}_{2,p} \| = 1,$$

$$\varepsilon_2(\{e^{ikt}\} \subset (H^n_p)^*) = \| I^{(n)}_{p,2} \| \sim n^{1/p - 1/2},$$

$$\varepsilon_2(\{e^{ikt}\} \subset H^n_\infty) = \| I^{(n)}_{2,\infty} \| \sim \sqrt{n},$$

$$\varepsilon_2(\{e^{ikt}\} \subset X^*) \leqq \varepsilon_2(\{e^{ikt}\} \subset L_1) = \| I^{(n)}_{2,1} \| = 1,$$

$$\mathbf{E}\left\| \sum g_k e^{ikt} \right\|_{H^n_p} \sim \mathbf{E}\left\| \sum g_k e^{ikt} \right\|_{(H^n_p)^*} \sim \sqrt{n},$$

and

$$\mathbf{E}\left\| \sum g_k e^{ikt} \right\|_{H^n_\infty} \sim \sqrt{n \log n},$$

$$\mathbf{E}\left\| \sum g_k e^{ikt} \right\|_{X^*} \leqq \mathbf{E}\left\| \sum g_k e^{ikt} \right\|_{L_1} \sim \sqrt{n},$$

so on using Theorem 3.3 we get

$$\mathcal{F}(H^n_p, H^n_\infty, X) \leqq c_p \sqrt{n}.$$

Since H^n_∞ contains $l_\infty^{[n/2]}$ uniformly, using the fact that L_p has cotype 2 if $1 \leqq p \leqq 2$, the lower estimate follows from

$$\mathcal{F}(H^n_p, H^n_\infty, X) \geqq d(H^n_p, H^n_\infty) \geqq c_p\beta_2(H^n_\infty) \geqq c_p\beta_2(l_\infty^{[n/2]}) \sim \sqrt{n}. \tag{9}$$

(ii) If $1 < p \leqq 2$, consider the factorization of H^n_p

$$H^n_p \xrightarrow[I^{(n)}_{p,1}]{} H^n_1 \xrightarrow[j]{} Y \xrightarrow[R|_Y]{} H^n_p$$

where j is the inclusion and R is the operator from L_1 to H^n_p defined by

$$R(f) = \frac{1}{2\pi} \sum e^{ikt} \int_0^{2\pi} f(s) e^{-iks} ds.$$

Identifying $(H^n_p)^*$ with $H^n_{p^*}$ $(1/p + 1/p^* = 1)$, we obtain

$$\|R\| = \|R^*\| \leqq c_p \|I^{(n)}_{p^*,\infty}\| \sim n^{1/p^*}$$

so $d(H^n_p, H^n_1) \leqq \mathscr{F}(H^n_p, H^n_1, Y) \leqq c_p n^{1/p^*}$.

Conversely, since H^n_1 contains $l_1^{[n/2]}$ uniformly and L_p has type p if $1 < p \leqq 2$, it follows that

$$d(H^n_p, H^n_1) \geqq \alpha_p(H^n_1)/\alpha_p(H^n_p) \geqq c_p \alpha_p(l_1^{[n/2]}) \geqq c_p [n/2]^{1/p^*}.$$

If $2 \leqq p < \infty$ we apply the estimates of Lemmas 3.2 and 3.3 together with Theorem 3.4 to get in the same manner as in (i) the inequality $\mathscr{F}(H^n_p, H^n_1, Y) \leqq c_p\sqrt{n}$. On the other hand, since H^n_1 contains a uniformly complemented subspace of dimension $[\gamma n] = m$ uniformly isomorphic to l_1^m (for $0 < \gamma < 1$ independent of n), therefore $(H^n_1)^*$ contains l_∞^m uniformly, and so identifying $(H^n_p)^*$ with a subspace of L_{p^*} and using the fact that every operator from l_∞^m to L_{p^*} is 2-summing, we obtain

$$\begin{aligned} d(H^n_1, H^n_p) &= d((H^n_1)^*, (H^n_p)^*) \\ &\geqq c_p \inf\{d(Z, l_\infty^m); Z \subset L_{p^*}, \dim Z = m\} \\ &\geqq cc_p \inf\{\pi_2(Z); Z \subset L_{p^*}, \dim Z = m\} \\ &\sim \sqrt{n} \end{aligned}$$

since $\pi_2(Z) = \sqrt{\dim Z}$ for every Banach space Z [6]. □

THEOREM 3.5. *If $\{p, q\} = \{1, \infty\}$ and X_q is any space satisfying $H^n_q \subseteq X_q \subseteq L_q$, then*

$$c\sqrt{n} \leqq d(H^n_1, H^n_\infty) \leqq \mathscr{F}(H^n_p, H^n_q, X_q) \leqq d\sqrt{n \log n}$$

for all integers $n \geqq 2$.

PROOF. The lower estimate follows from inequality (9) above. The upper estimate follows from using the estimates of Lemmas 3.1 and 3.2 together with Theorem 3.3. □

REMARKS. (1) It is unknown whether any of the inequalities in Theorem 3.5 is sharp.

If however the dimension of H^n_q is increased to $a \cdot n$ in Theorem 3.5 then $\mathscr{F}(H^n_p, H^{an}_q) \sim \sqrt{n}$, where $2 \leqq a < \infty$ is independent of n. Indeed H^{2n}_∞ contains l^n_∞ uniformly complemented, hence it suffices to prove $\sqrt{n} \geqq d(l^n_\infty, H^n_1)$. But if $T: H^n_1 \to l^n_\infty$ is the map defined by $T(e^{ikt}) = e_k$ $(1 \leqq k \leqq n)$, then $\|T\| = \|T^*\| = 1$ and

$$\| T^{-1} \| = \| T^{-1*} \| = \sup \left(\sum_1^n | t_k | / \left\| \sum_1^n t_k e^{ikt} \right\|_{(H_1^n)^*} \right)$$

$$\leqq \sup \left(\sum | t_k | / \sqrt{\sum | t_k |^2} \right) = \sqrt{n}.$$

Similarly H_1^{an} contains l_1^n uniformly complemented for some $\infty > a \geqq 2$ independent of n, hence it suffices to prove $d(l_1^n, H_\infty^n) \leqq \sqrt{n}$ to imply that $\mathscr{F}(H_\infty^n, H_1^{an}) \leqq c\sqrt{n}$. But the proof is identical for this case too. The facts that $\mathscr{F}(H_p^n, H_q^{an}) \geqq c\sqrt{n}$ if $\{p, q\} = \{1, \infty\}$ are proved as in Theorems 3.4 and 3.5.

(2) It is easy to see that

$$c_1 \sqrt{\log n} \leqq d(H_1^n, l_1^n) \leqq c_2 \log n.$$

It would be interesting to know the exact values for this quantity.

§4. A remark on absolutely summing operators from H_∞

In this section, which is not directly connected with the preceding sections, we bring an observation which answers problem 3.2 in [18].

Theorem 2.4 in [18] asserts that for $1 < p < \infty$ every p-absolutely summing operator from A is strictly p-integral and there is a constant C_p such that

$$i_p(T) \leqq C_p \pi_p(T)$$

for all such T. Problem 3.2 asks whether every p-a.s. T from H_∞ is p-integral.

PROPOSITION 4.1. *For every Banach space E and $T \in \pi_p(H_\infty, E)$ $(1 < p < \infty)$ T is p-integral and*

$$i_p(T) \leqq C_p \pi_p(T).$$

(C_p is the same constant as above.)

PROOF. For Banach spaces E, F and $T: E \to F$ a linear operator, we define $(i_p/\pi_p)(T)$ to be

$$\frac{i_p}{\pi_p}(T) = \sup i_p(ST);$$

the sup is taken over all Banach spaces G and operators $S: F \to G$ with $\pi_p(S) \leqq 1$. i_p and π_p are perfect ideal norms, also π_p is semi-tensorial (see [22] for definition) hence by [22] proposition 2.7 we conclude that i_p/π_p is a perfect ideal norm.

From [20] it follows now that $i_p/\pi_p = (i_p/\pi_p)''$, i.e. $(i_p/\pi_p)(T) = (i_p/\pi_p)(T^{**})$ for all operators $T: E \to F$.

H_∞ is a 1-complemented subspace of A^{**}; it is enough, therefore, to show that

$$\frac{i_p}{\pi_p}(\mathrm{Id}_{A^{**}}) \leqq C_p.$$

By theorem 2.4 [18] we have

$$\frac{i_p}{\pi_p}(\mathrm{Id}_A) \leqq C_p$$

hence

$$\frac{i_p}{\pi_p}(\mathrm{Id}_{A^{**}}) = \frac{i_p}{\pi_p}(\mathrm{Id}_A^{**}) = \left(\frac{i_p}{\pi_p}\right)''(\mathrm{Id}_A) = \frac{i_p}{\pi_p}(\mathrm{Id}_A) \leqq C_p. \qquad \square$$

References

1. Y. Benyamini and Y. Gordon, *Random factorization of operators between Banach spaces*, J. Analyse Math. **39** (1981), 45–74.

2. J. Bourgain, *A remark on finite dimensional P_λ spaces*, Studia Math., to appear.

3. E. W. Cheney and K. H. Price, *Minimal projections*, in *Approximation Theory Proc. Lancaster 1969 Symp.*, Academic Press, 1970.

4. P. Duren, *Theory of H^p Spaces*, Academic Press, 1970.

5. X. M. Fernique, *Regularité des trajectories des fonctions aléatoires Gaussiens*, Lecture Notes in Math. **480**, Springer Verlag, 1975, pp. 1–96.

6. D. J. H. Garling and Y. Gordon, *Relations between some constants associated with finite dimensional Banach spaces*, Israel J. Math. **9** (1971), 346–361.

7. Y. Gordon, *p-local unconditional structure of Banach spaces*, Compositio Math. **41** (1980), 189–205.

8. Y. Gordon and D. R. Lewis, *Absolutely summing operators and local unconditional structures*, Acta Math. **133** (1974), 27–48.

9. Y. Gordon, D. R. Lewis and J. R. Retherford, *Banach ideals of operators with applications*, J. Functional Analysis **14** (1973), 85–129.

10. V. I. Gurarii, M. I. Kadec and V. I. Macaev, *Distances between finite dimensional analogues of L_p spaces*, Math. Sb. **70** (112) (1966), 481–489.

11. W. Johnson and G. Schechtman, *On subspaces of L_1 with maximal distance to Euclidean spaces*, Proceeding of University of Iowa, to appear.

12. S. V. Kisliakov, *What is needed for a 0-absolutely summing operator to be nuclear?* Complex Analysis and Spectral Theory Seminar, Leningrad 1969/80, Lecture Notes in Math. **864**, Springer-Verlag.

13. S. Kwapien, *On a theorem of L. Schwartz and its applications to absolutely summing operators*, Studia Math. **38** (1970), 193–201.

14. S. Kwapien and A. Pelczynski, *Remarks on absolutely summing translation invariant operators from the disc algebra and its dual into a Hilbert space*, Michigan Math. J. **25** (1978), 173–181.

15. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, Vols. I and II, Springer Verlag, 1977 and 1979.

16. B. Maurey, *Théorèmes de factorisation pour les opérateurs linéaires à valeurs dans les espaces L^p*, Soc. Math. France, Asterisque **11**, 1974.

17. B. Maurey, *Isomorphismes entre espaces H_1*, Acta Math. **145** (1980), 79–120.

18. A. Pelczynski, *Banach spaces of analytic functions and absolutely summing operators*, AMS Regional Conference Series in Math., **30**, Providence, 1977.

19. A. Persson and A. Pietsch, *p-nukleare und p-integrale operatoren in Banach Räumen*, Studia Math. **33** (1969), 21–62.

20. A. Pietsch, *Adjungierte normierten operatoren-ideale*, Math. Nachr. **48** (1971), 189–211.

21. A. Pietsch, *Operator Ideals*, North-Holland, 1980.

22. S. Reisner, *On Banach spaces having the property* G.L., Pacific J. Math. **83** (1979), 505–521.

23. L. Schwartz, *Applications du théorème de dualité sur les applications p-radonifiantes*, C. R. Acad. Sci. Paris **268** (1969), 1612–1615.

24. P. Wojtaszczyk, *The Franklin system is an unconditional basis in* H_1, preprint.

25. A. Zygmund, *Trigonometric Series I, II*, Cambridge University Press, London and New York, 1959.

TEXAS A&M UNIVERSITY
COLLEGE STATION, TX 77843 USA
AND
TECHNION — ISRAEL INSTITUTE OF TECHNOLOGY
HAIFA, ISRAEL